MATH-F410 – REP. DES GROUPES & APP. A LA PHYS.

Solutions des exercices 2023-2024

Séance 2 : théorème de Burnside et orthogonalité des caractères

1. Irreducible representations of S_3 and S_4

Section a)

It is shown in the lecture notes that any two cycles of equal length are conjugate. We can produce a refined version of this result which says that two permutations are conjugate if and only if their decomposition into disjoint cycles produces the same number of cycles of a given length (try to prove this, the technique is totally analogous to the one used in the lecture notes, just generalizing a bit the notation to include more cases!). For S_4, we will have then the following conjugacy classes:

$$\mathcal{C}(S_4) = \{[e], [(1\,2)], [(1\,2\,3)], [(1\,2\,3\,4)], [(1\,2)(3\,4)]\} \ .$$

You can compute how many elements there are in each conjugacy class (the total is obviously $4! = 24$):

- $[e]$ has only one element, the identity.
- $[(1\,2)]$ has $\binom{4}{2} = 6$ elements, all of which are 2-cycles (transpositions).
- $[(1\,2\,3)]$ has 8 elements, all of which are 3-cycles.
- $[(1\,2\,3\,4)]$ has $3! = 6$ elements, all of which are 4-cycles.
- $[(1\,2)(3\,4)]$ has 3 elements, all of which are products of two 2-cycles.

Section b)

Take $V = \{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$. This is a union of conjugacy classes (the first and the last one in the previous list), and therefore it is invariant under conjugation. If it is a subgroup of S_4, it will be a normal subgroup. We just check it satisfies the conditions to be a subgroup:

- $e \in V$ clearly.
- All elements $a \in V$ satisfy $a^2 = e$. Thus, $a^{-1} = a$ for all $a \in V$; and the inverse of any element in V is thus also in V.
- The final step is to check that the product is closed within V. Notice that the non-trivial elements are of the form $(1\,a)(b\,c)$. Multiplying two which are different

(we already know that squares are $e \in V$) can be done in general as (reorganize the separate transpositions to write any product of distinct elements as follows):

$$(1\,a)(b\,c)(1\,b)(a\,c) = (1\,c)(a\,b)\ .$$

The product is thus closed within V. If this abstract argument does not convince you, just do all the 6 non-trivial products!

Section c)

The quotient S_4/V defines a group because V is a normal subgroup. The number of elements is $|S_4|/|V| = 6$, so it coincides with the number of elements of S_3. This is a good sign, but we still have to construct the isomorphism between S_4/V and S_3. The key for this is understanding a bit the cosets:

$$(1\,2)V = \{(1\,2), (3\,4), (1\,3\,2\,4), (1\,4\,2\,3)\} = (3\,4)V\ .$$

Notice that multiplying by the two complementary transpositions $(1\,2)$ and $(3\,4)$ produces the same coset. Also, the 4-cycles are produced as part of these cosets. This is true for the other transpositions as well:

$$\begin{aligned}(1\,3)V &= \{(1\,3), (2\,4), (1\,2\,3\,4), (1\,4\,3\,2)\} = (2\,4)V\ ,\\ (2\,3)V &= \{(2\,3), (1\,4), (1\,2\,4\,3), (1\,3\,4\,2)\} = (1\,4)V\ .\end{aligned}$$

These account for all the 12 odd permutations of S_4. The remaining 12 even ones are in the trivial coset $eV = V$ and in the ones obtained multiplying by 3-cycles:

$$\begin{aligned}(1\,2\,3)V &= \{(1\,2\,3), (1\,3\,4), (2\,4\,3), (1\,4\,2)\}\ ,\\ (1\,3\,2)V &= \{(1\,3\,2), (2\,3\,4), (1\,2\,4), (1\,4\,3)\}\ .\end{aligned}$$

Notice that we have chosen a clever way to label the cosets. You can call it gV for any g in that particular coset, but we have chosen to do it with $g \in S_3$ so that the isomorphism suggests itself:

$$F : S_3 \longrightarrow S_4/V\ , \qquad F(\sigma) = \sigma V\ \forall \sigma \in S_3\ .$$

This maps S_3 to $S_4/V = \{V, (1\,2)V, (1\,3)V, (2\,3)V, (1\,2\,3)V, (1\,3\,2)V\}$ in a one-to-one way (this is what we have just shown above), and respecting the multiplication rule. This is just a consequence of the multiplication rule for cosets, *e.g.*:

$$(1\,2)V\ (1\,2\,3)V = (1\,2)(1\,2\,3)V = (2\,3)V\ .$$

Section d)

Keeping the notation used until now (cosets are gV, conjugacy classes $[g]$), we are inducing an S_4 representation $\tilde{T}$ from an S_3 one T as $\tilde{T}(g) = T(gV)$.[1] Is this a representation? It

[1] Technically, the S_3 element is $F^{-1}(gV)$, but since F is an isomorphism we will not try to be more pedantic than strictly needed.

is indeed a linear map acting on the same representation space as that of T, but we have to check whether it respects the product rule:

$$\tilde{T}(gg') = T((gg')V) = T((gV)(g'V)) = T(gV)T(g'V) = \tilde{T}(g)\tilde{T}(g') ,$$

so it is indeed a representation of S_4. To check it is irreducible if T is irreducible we will use the criterion based on the value of the character. Representations are irreducible if and only if the norm of their characters (computed as functions of $L^2(G)$) is 1. So, if T is irreducible:

$$1 = (\chi_T, \chi_T)_M = \frac{1}{|S_3|} \sum_{g \in S_3} \chi_T^\star(g)\chi_T(g) .$$

On the other hand, for $\tilde{T}$, we will split the sum over S_4 elements into the different cosets computed before:

$$\begin{aligned}
(\chi_{\tilde{T}}, \chi_{\tilde{T}})_M &= \frac{1}{|S_4|} \sum_{\tilde{g} \in S_4} \chi_{\tilde{T}}^\star(\tilde{g})\chi_{\tilde{T}}(\tilde{g}) = \frac{1}{|S_4|} \sum_{\tilde{g} \in S_4} \chi_T^\star(\tilde{g}V)\chi_T(\tilde{g}V) \\
&= \frac{1}{|S_4|} \sum_{g \in S_3} \sum_{h \in V} \chi_T^\star(ghV)\chi_T(ghV) \\
&= \frac{|V|}{|S_4|} \sum_{g \in S_3} \chi_T^\star(gV)\chi_T(gV) = \frac{|V||S_3|}{|S_4|}(\chi_T, \chi_T)_M = 1 ,
\end{aligned}$$

so $\tilde{T}$ is indeed irreducible. In the second line we wrote any $\tilde{g} \in S_4$ as $\tilde{g} = gh$ with $g \in S_3$, $h \in V$. In the third one we used $hV = V$ for any $h \in V$, so each sum over $h \in V$ just produces $|V|$ equal terms. The final step is just the relation between the orders of the different groups involved, $|S_3| = |S_4|/|V|$.

Section e)

The idea here is to use S_3 irreducible representations and the result from the previous section to obtain S_4 irreps. So we have to find three inequivalent irreps of S_3. Recall that the number of inequivalent irreps of a finite group G coincides with the number of its conjugacy classes, and, much like in section a), S_3 has three:

$$\mathcal{C}(S_3) = \{[e], [(1\,2)], [(1\,2\,3)]\} .$$

The dimensions of the representations are obtained by Burnside's theorem: $6 = 1 + n_1^2 + n_2^2$ (the 1 is the trivial representation). The only possible solution is $n_1 = 1$, $n_2 = 2$, therefore we must find, in addition to the trivial representation, another 1-dimensional one and a 2-dimensional one. For S_3, this is relatively easy:[2]

[2] The procedure we will follow is not systematic. There is a systematic one to find all the irreducible representations of a finite group, which consists in starting from the regular representation and decomposing into irreducible representations. This can be done with the techniques explained in chapter 3 of the lecture notes, which show that the regular representation T_R decomposes as $T_R = T_0 \oplus T_1 \oplus 2T_2$, with T_0 being the trivial representation.

- The representation T_1 is 1-dimensional, so it represents each group element as a number $T_1(g) = \lambda_g \in \mathbb{C}$ for any $g \in S_3$. As in any representation, $\lambda_e = 1$, and not all elements are represented by 1 (otherwise the representation would be the trivial one). Notice then that $\lambda^2_{(1\,2)}$ must be 1 due to $(1\,2)(1\,2) = e$, so $\lambda_{(1\,2)} = \pm 1$. Any other 2-cycle is related to this one by conjugation and, being the representation 1-dimensional (so that everything commutes), this necessarily implies $\lambda_{(1\,2)} = \lambda_{(1\,3)} = \lambda_{(2\,3)} = \pm 1$. We have also that $\lambda_{(1\,2\,3)} = \lambda_{(1\,3)}\lambda_{(1\,2)} = 1$ (and same for $\lambda_{(1\,3\,2)}$, which is the inverse). To have a representation which is not trivial, we must pick $\lambda_{(1\,2)} = -1$. Let's recap and write the full representation $T_1(g) = \lambda_g$:

  $$\lambda_e = 1 \ , \qquad \lambda_{(1\,2)} = \lambda_{(1\,3)} = \lambda_{(2\,3)} = -1 \ , \qquad \lambda_{(1\,2\,3)} = \lambda_{(1\,3\,2)} = 1 \ .$$

  This is the **sign representation**. It exists for any permutation group, it is 1-dimensional, and it assigns to any permutation its sign (its determinant thought as a permutation matrix).

- The 2-dimensional T_2 representation is a bit more difficult to obtain. However, we can already answer some questions about it in a systematic way. Knowledge of the other two representations implies knowledge of their characters (χ_i is the character of the representation T_i, with T_0 being the trivial one):

	$[e]$	$[(1\,2)]$	$[(1\,2\,3)]$
χ_0	1	1	1
χ_1	1	-1	1
χ_2	?	?	?

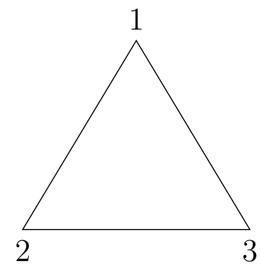
  We can fill in the last row (*i.e.*, we can obtain the character of the T_2 representation) by orthonormality conditions with respect to the mean inner product in $L^2(S_3)$:

  $$\begin{aligned}(\chi_i, \chi_j)_M &= \frac{1}{|S_3|} \sum_{g \in S_3} \chi_i^\star(g)\chi_j(g) \\ &= \frac{1}{6}\left[\chi_i^\star(e)\chi_j(e) + 3\chi_i^\star(1\,2)\chi_j(1\,2) + 2\chi_i^\star(1\,2\,3)\chi_j(1\,2\,3)\right] = \delta_{i,j} \ .\end{aligned}$$

  The result is (the character of e is always the dimension of the representation):

	$[e]$	$[(1\,2)]$	$[(1\,2\,3)]$
χ_0	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

  The precise form of the representation cannot be deduced from here. However, as we said, S_3 is simple enough to guess it. If you recall $S_3 \cong D_3$, you can think of S_3 as the symmetry transformations of an equilateral triangle. This is a 2-dimensional representation, which is precisely what we are after. In this language, $T_2(1\,2\,3)$ and $T_2(1\,3\,2)$ are rotations of $2\pi/3$ and $-2\pi/3$ respectively, and the transpositions are reflections about symmetry axes. If you picture the triangle as follows:



then it is easy to write the reflection $T_2(2\,3) = \mathrm{diag}(-1,1)$. From this and the rotation matrices it is easy to write the full representation:

$$T_2(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad T_2(1\,2) = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix} \qquad T_2(1\,3) = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$T_2((2\,3)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad T_2(1\,2\,3) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \quad T_2(1\,3\,2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

You can check that the characters (traces) match the ones written in the previous table, and that the group multiplication law is respected.

This is all for S_3 irreps. Following the definition in the previous section, the three irreps become also irreps os S_4 as:

$$\tilde{T}_i(g) = T_i(gV) \quad \forall g \in S_4 \ .$$

The matching to S_3 elements is done through the cosets written in section c). The trivial representation of S_3 induces the trivial representation of S_4, and the sign one also maps to the sign representation of S_4.

Section f)

We use Burnside's theorem and the already known S_4 irreps:

$$4! = 24 = 1 + 1 + 4 + n_3^2 + n_4^2 \ ,$$

where we note that we know the total number of irreps is 5, since that is the number of conjugacy classes of S_4. There is a single solution for this equation with integer variables, $n_3 = n_4 = 3$. There are then two extra irreps of S_4, T_3 and T_4, both 3-dimensional.

Section g)

The table of characters is computed much in the same way we did for S_3. So far we know:

	$[e]$	$[(1\,2)]$	$[(1\,2\,3)]$	$[(1\,2\,3\,4)]$	$[(1\,2)(3\,4)]$
χ_0	1	1	1	1	1
χ_1	1	-1	1	-1	1
χ_2	2	0	-1	0	2
χ_3	3	?	?	?	?
χ_4	3	?	?	?	?

Orthonormality relations will not be enough to obtain the two remaining rows. One way to proceed in this situation is to come up with some representation (not necessarily irreducible), find the components in terms of the known irreps, and hope that the remaining piece gives us information about some up to now unknown representation. This is easier to understand in a practical computation. A representation (reducible) which is always at hand for S_n is the n-dimensional one which writes the permutations as matrices. Call it T_P. For S_4, for example:

$$T_P(1\,2\,3\,4) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad T_P(2\,3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \ldots$$

It is clear that the trace of these matrices is equal to the number of elements left invariant by the corresponding permutation. Thus:

	$[e]$	$[(1\,2)]$	$[(1\,2\,3)]$	$[(1\,2\,3\,4)]$	$[(1\,2)(3\,4)]$
χ_P	4	2	1	0	0

Using the product in $L^2(S_4)$ for the characters:

$$(\chi_i, \chi_j)_M = \frac{1}{|S_4|} \sum_{g \in S_4} \chi_i^\star(g)\chi_j(g) = \frac{1}{24} \left[\chi_i^\star(e)\chi_j(e) + 6\chi_i^\star(1\,2)\chi_j(1\,2) + 8\chi_i^\star(1\,2\,3)\chi_j(1\,2\,3) \right.$$
$$\left. + 6\chi_i^\star(1\,2\,3\,4)\chi_j(1\,2\,3\,4) + 3\chi_i^\star((1\,2)(3\,4))\chi_j((1\,2)(3\,4))\right] ,$$

we compute the components of T_P along the known irreps:

$$(\chi_0, \chi_P)_M = 1 , \qquad (\chi_1, \chi_P)_M = 0 , \qquad (\chi_2, \chi_P)_M = 0 .$$

We thus discover a new representation by substracting the trivial one, let's call it T_3 (anticipating it will be irreducible):

	$[e]$	$[(1\,2)]$	$[(1\,2\,3)]$	$[(1\,2\,3\,4)]$	$[(1\,2)(3\,4)]$
χ_3	3	1	0	-1	-1

Computing the norm, we see that $(\chi_3, \chi_3)_M = 1$, so it is irreducible. We fill in the remaining row in the table by orthonormality, which is now enough:

	$[e]$	$[(1\,2)]$	$[(1\,2\,3)]$	$[(1\,2\,3\,4)]$	$[(1\,2)(3\,4)]$
χ_0	1	1	1	1	1
χ_1	1	-1	1	-1	1
χ_2	2	0	-1	0	2
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	1	-1

Actually, you can even fill in this last row more quickly by noting that it is just the product $T_1 \otimes T_3$ (but tensor products are not introduced until the next exercise sheet!).

2. $\mathbb{Z}_4$ irreducible representations

Section a)

The group described in the question is clearly $\mathbb{Z}_4$. Since it is Abelian, the number of conjugacy classes is equal to the number of elements (see next exercise), so there are 4. This is also the number of inequivalent irreps, all of them of dimension 1.

Section b)

We will present the argument for $\mathbb{Z}_4$, but it can be easily generalized to any $\mathbb{Z}_n$. Call a an element of order 4, so that $\mathbb{Z}_4 = \{a, a^2, a^3, a^4 = e\}$ (in the framework of this exercise, a could be a rotation by $\pi/2$). One of the inequivalent irreps is the trivial representation, call it T_0, for which $T_0(a^n) = 1$ for any n mod 4. Any other representation must satisfy:

- $T(e) = 1$
- $T(a) = x \neq 1$
- $x^4 = T(a)^4 = T(a^4) = T(e) = 1.$

The representations are then generated by $T(a) = x$ a fourth root of unity. Since the characters of a 1-dimensional representation are just the values of the representatives, and since in an Abelian group the conjugacy classes have a single element, we will directly present the table of characters, which is equivalent to giving the representatives of all elements:

	e	a	a^2	a^3
χ_0	1	1	1	1
χ_1	1	i	-1	$-i$
χ_2	1	-1	1	-1
χ_3	1	$-i$	-1	i

Section c)

The regular representation of $\mathbb{Z}_4$ was obtained in the first exercise sheet:

$$T_R(e) = \mathbb{I} \ , \qquad T_R(a) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ ,$$

$$T_R(a^2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \ , \qquad T_R(a^3) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ .$$

The characters of this regular representation are:

	e	a	a^2	a^3
χ_R	4	0	0	0

It is evident then that $\chi_R = \chi_0 + \chi_1 + \chi_2 + \chi_3$. This is a verification of the result presented in the lectures: the regular representation contains every inequivalent irrep, with a degeneracy equal to the dimension of the irrep (in this case, all of them are 1-dimensional).

3. Irreducible representations of Abelian groups

There are as many irreps as conjugacy classes in a finite group. For an Abelian group G, consider an element $h \in G$. The corresponding conjugacy class has only one element, since $ghg^{-1} = h$ for any $g \in G$. Thus, we have $|G| = n$ conjugacy classes, each one with a single element. Burnside's theorem then with n independent irreps gives:

$$n = \sum_{i=1}^{n} n_i^2 \ .$$

The only solution is $n_i = 1 \ \forall i = 1, \ldots, n$. We have n inequivalent irreps of dimension 1 for any Abelian group.